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A Remark on Association Schemes with Two P-polynomial Structures

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Association schemes with two different P-polynomial structures have been classified by Gardiner, and by Bannai and Bannai. In this note we derive very restrictive conditions for one subcase in this classification.

Gardiner [5] and Bannai and Bannai [1] classified symmetric association schemes $(X, \{R_0, \ldots, R_d\})$ such that for (at least) two different choices of i, the graph (X, R_i) is distance-regular with diameter d. It turns out that except in the case of a polygon there are never more than two choices for i. Let one of the choices be $i = 1$, and let the ordering of the relations be such that R_j is the relation of being at distance j in the graph $\Gamma = (X, R_1)$ $(0 \leqslant j \leqslant d)$. Then for the other choice we have $i \in \{2, d-1, d\}$, and the ordering of the relations according to distance in the graph $\Delta = (X, R_i)$ is one of the following (cf. Bannai and Ito [2], Theorem III.4.2, p. 241):

(I) $R_0, R_2, R_4, R_6, \ldots, R_5, R_3, R_1$;
(II) $R_0, R_d, R_1, R_{d-1}, R_2, R_{d-2}, R_3, \ldots$;
(III) $R_0, R_d, R_2, R_{d-2}, R_4, R_{d-4}, \ldots, R_{d-5}, R_5, R_{d-3}, R_3, R_{d-1}, R_1$;
(IV) $R_0, R_{d-1}, R_2, R_{d-3}, R_4, R_{d-5}, \ldots, R_5, R_{d-4}, R_3, R_{d-2}, R_1, R_d$.

Types I and II are duals of each other (if Δ is of type II w.r.t. Γ, then Γ is of type I w.r.t. Δ); types III and IV are self-dual.

For types I and II, examples are known derived from the Odd graphs, the folded $(2d+1)$-cubes, or from coset graphs related to the binary Golay code $(d=3)$. (Of course, all strongly regular graphs give examples with $d = 2$ of both I, II and III.)

For type IV, examples are known derived from the d-cubes, from a coset graph related to the binary Golay code $(d = 6)$, the dodecahedron $(d = 5)$, from Hadamard graphs $(d = 4)$, from the Wells graph $(d = 4)$, and from Taylor graphs $(d = 3)$.

For type III, the only examples known are derived from square 2-designs and have $d = 3$. We expect that it will be possible by use of the results in this note to rule out this case when d is large (say, $d \geqslant 7$).

For details on association schemes and distance-regular graphs, see Bannai and Ito [2] and Brouwer, Cohen and Neumaier [3]; we shall refer to these books and [BI] and [BCN]. We use the standard notation $a_i = p^i_{1,i}$, $b_i = p^i_{1,i+1}$, $c_i = p^i_{1,i-1}$, $k_i = p^0_{i,i}$, $k = k_1$, $\lambda = a_1$, $\mu = c_2$ for the graph Γ, and use notation such as $\mu_\Delta = c_2(\Delta)$ for the graph Δ.

THEOREM. *Let Γ, Δ be two P-polynomial structures on the same association scheme, with distance correspondence*

Γ	0	1	2	3	4	$\cdots$	$d-3$	$d-2$	$d-1$	d
Δ	0	d	2	$d-2$	4	$\cdots$	$d-3$	3	$d-1$	1

and $d \geqslant 5$. Assume that $k > 2$. Then:
(*i*) $k(b_{d-1} - 1) = (k_d - 1)\mu$;
(*ii*) *if $d \geqslant 7$, then d is odd and Γ and Δ are bipartite*;

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(iii) if $d \geq 6$ or Γ is not bipartite, then

$$\mu\mu_\Delta = (b_{d-1} - 1)^2 + 1, \tag{1}$$

$$k - 1 = (b_{d-1} - 1)(b_{d-1} - 1 + \mu), \qquad k_d - 1 = (b_{d-1} - 1)(b_{d-1} - 1 + \mu_\Delta), \tag{1a,b}$$

$$k = \mu(b_{d-1} - 1 + \mu_\Delta), \qquad k_d = \mu_\Delta(b_{d-1} - 1 + \mu), \tag{1c,d}$$

$$b_{d-1} = \tfrac{1}{2}(\mu + \mu_\Delta - 3). \tag{2}$$

PROOF. By [BI], Theorem III.4.2 or [BCN], 4.2.15(iii) we have: if d is odd, then $a_i = 0$ for $i \neq (d+1)/2$; if d is even, then $a_i \neq 0$ if and only if $i \in \{\tfrac{1}{2}d, \tfrac{1}{2}d + 1\}$; $p^d_{j,d} = 0$ for $j \neq 0, 2$. Now fix vertices α, $\beta \in \Gamma$ with $d(\alpha, \beta) = d$, and count paths $\beta \sim \gamma \sim \delta$, where $d(\alpha, \delta) = d$. We find (i). This part is due to J. T. M. van Bon. By Brouwer and Lambeck [4] or [BCN], Proposition 5.5.4(iii), we have

$$\text{if } a_i \neq 0 \text{ then } k \leq 2a_i + \frac{a_{i+1}}{a_i} b_i + \frac{a_{i-1}}{a_i} c_i. \tag{3}$$

If d is odd, $d = 2e - 1$, and $a_e \neq 0$, then it follows that $a_e \geq \tfrac{1}{2}k$. If $d = 2e$ is even, then we obtain

$$k \leq 2a_e + \frac{a_{e+1}}{a_e} b_e \qquad \text{and} \qquad k \leq 2a_{e+1} + \frac{a_e}{a_{e+1}} c_{e+1}. \tag{4,5}$$

For large d and non-bipartite Γ, it follows that c_3 is rather small.

If $d = 2e - 1 \geq 7$, $a_e \geq 0$, then we have $c_3 \leq c_e$ (since $e \geq 3$ and c_i increases monotonically with i), and $c_3 \leq b_e$ (since $c_i \leq b_{d-i}$), and $\tfrac{1}{2}k \leq a_e$ (see above), and by adding these three inequalities (using that $k = a_i + b_i + c_i$) it follows that $c_3 \leq \tfrac{1}{4}k$.

If $d = 2e \geq 8$, then if $a_e \leq a_{e+1}$ we find $k \leq 2a_{e+1} + c_{e+1}$, so $c_3 \leq b_{e+1} \leq a_{e+1}$, but also $c_3 \leq c_{e+1}$, so $c_3 \leq \tfrac{1}{3}k$. On the other hand, if $a_{e+1} \leq a_e$, then we find (using only $d \geq 6$) $k \leq 2a_e + b_e$, so $c_3 \leq c_e \leq a_e$, but also $c_3 \leq b_e$, so $c_3 \leq \tfrac{1}{3}k$ also in this case.

For smaller d we obtain slightly weaker estimates: if $d = 6$, then $2c_3 \leq c_3 + b_3 < k$, so $c_3 < \tfrac{1}{2}k$; if $d = 5$ and $a_3 \neq 0$, then again $c_3 < \tfrac{1}{2}k$, since $a_3 \geq \tfrac{1}{2}k$.

By [BCN], Corollary 5.8.2, we have

$$k - 2 \geq (\mu - 1)\left(\frac{\mu(k-2)}{c_3 - 1} - \mu + 2\right). \tag{6}$$

But if $c_3 \leq \tfrac{1}{3}k$, then (6) yields $k - 2 > (\mu - 1)(3\mu - \mu + 2)$, i.e.

$$k > 2\mu^2. \tag{7a}$$

If only $c_3 < \tfrac{1}{2}k$, then (6) yields $k - 2 > (\mu - 1)(2\mu - \mu + 2)$, i.e.

$$k > \mu^2 + \mu. \tag{7b}$$

Now, let us compute μ_Δ. We find (using (i), and $k_2\mu = kb_1$ and $b_1 = k - 1$):

$$\mu_\Delta = p^2_{d,d} = \frac{k_d}{k_2} p^d_{2,d} = \frac{k_d(k_d - 1)}{k_2} = \frac{k_d(b_{d-1} - 1)}{k - 1}$$
$$= \frac{(k(b_{d-1} - 1) + \mu)(b_{d-1} - 1)}{\mu(k-1)} = \frac{1}{\mu}\left((b_{d-1} - 1)^2 + \frac{(b_{d-1} - 1)(b_{d-1} - 1 + \mu)}{k - 1}\right). \tag{8}$$

If we choose Γ in the pair (Γ, Δ) so that $k_\Gamma \geq k_\Delta$, i.e. $k \geq k_d$, then (i) yields $\mu > b_{d-1} - 1$, i.e. $\mu \geq b_{d-1}$. Since μ_Δ is integral, we have $(k-1) \mid (b_{d-1} - 1)(b_{d-1} - 1 + \mu)$, so

$$k - 1 \leq (b_{d-1} - 1)(b_{d-1} - 1 + \mu) \leq (\mu - 1)(2\mu - 1) < 2\mu^2. \tag{9}$$

(Note that $b_{d-1} \neq 1$ by (i).) But (7a) and (9) contradict each other, so either Γ and Δ are bipartite of odd diameter (note that Δ is bipartite if and only if Γ is), or $d \leq 6$. This proves (ii).

On the other hand, if $k-1 \neq (b_{d-1}-1)(b_{d-1}-1+\mu)$, then

$$k-1 \leq \tfrac{1}{2}(b_{d-1}-1)(b_{d-1}-1+\mu) < \mu^2. \tag{10}$$

But (7b) and (10) contradict each other, so this is possible only when $d=5$ and $a_3=0$. Assume that this is not the case; then we find

$$k-1=(b_{d-1}-1)(b_{d-1}-1+\mu), \tag{1a}$$

or, equivalently,

$$\mu\mu_\Delta=(b_{d-1}-1)^2+1. \tag{1}$$

Since this latter condition is symmetric in Γ and Δ (because $b_{d-1}(\Gamma)=p^{d-1}_{1,d}=b_{d-1}(\Delta)$), it does not depend on the assumption that $k \geq k_d$, and we also have

$$k_d-1=(b_{d-1}-1)(b_{d-1}-1+\mu_\Delta). \tag{1b}$$

Using (i), the equations (1c) and (1d) follow.

It remains to prove (2). Let us compute k_{d-1}:

$$k_{d-1}=\frac{kk_d}{b_{d-1}}=\frac{\mu\mu_\Delta(b_{d-1}-1+\mu)(b_{d-1}-1+\mu_\Delta)}{b_{d-1}}$$

This is an integer, and by (1) we have $\mu\mu_\Delta \equiv 2 (\operatorname{mod} b_{d-1})$, so $b_{d-1} \mid 2(\mu-1)(\mu_\Delta-1)$, and

$$b_{d-1} \mid 2(\mu+\mu_\Delta-3).$$

Assume again that $k \geq k_d$; then $\mu_\Delta < b_{d-1} \leq \mu$ (since $k > k_d$ by (i)).

Since $(b_{d-1}-1)(b_{d-1}-1+\mu)+1=k>\mu^2+\mu$ by (1a) and (7b), it follows that $b_{d-1}>\tau^{-1}\mu$, where τ is the golden ratio ($\tau^2=\tau+1$, $\tau \approx 1{\cdot}6$, $\tau^{-1} \approx 0{\cdot}6$). Thus, if we put

$$m=2(\mu+\mu_\Delta-3)/b_{d-1},$$

then $(m-2)\tau^{-1}\mu<(m-2)b_{d-1}<2\mu$, so $m<2(\tau+1)\approx 5{\cdot}2$; i.e.

$$m \in \{0, 1, 2, 3, 4, 5\}.$$

We want to show that $m=4$.

If $m=5$, then $b_{d-1}<\frac{2}{3}\mu$, so $\mu\mu_\Delta=(b_{d-1}-1)^2+1<\frac{4}{9}\mu^2$, so $\mu_\Delta<\frac{4}{9}\mu$, so $5b_{d-1}<(2+\frac{8}{9})\mu<3\mu$, so $k<k_{b-1}(\mu+b_{d-1})<\frac{24}{25}\mu^2$, a contradiction.

If $m=0$, then $\mu+\mu_\Delta=3$, $\mu_\Delta=1$, $\mu=2$, $b_{d-1}=2$, $k=4$, contradicting (7b).

If $1 \leq m \leq 3$, then

$$\tfrac{1}{4}(\mu+\mu_\Delta)^2 \geq \mu\mu_\Delta=(b_{d-1}-1)^2+1 \geq (\tfrac{2}{3}(\mu+\mu_\Delta-3)-1)^2+1,$$

so $\frac{7}{36}(\mu+\mu_\Delta)^2 \leq 4(\mu+\mu_\Delta)-10$, i.e. $\mu+\mu_\Delta \leq [144/7]=20$, hence $b_{d-1} \leq 10$.

If $b_{d-1}=10$, then $\mu\mu_\Delta=82$, but 41 is prime, which is impossible.

If b_{d-1} is odd, then m is even, so $m=2$, $b_{d-1}=\mu+\mu_\Delta-3$, but $b_{d-1} \leq \mu$, so $\mu_\Delta \leq 3$ and we find $\mu=5$, $\mu_\Delta=1$, $b_{d-1}=3$, $k=15$, contradicting (7b).

If $b_{d-1}=8$, then $\mu\mu_\Delta=50$, $4m=\mu+\mu_\Delta-3$, $\mu=10$, $\mu_\Delta=5$, $m=3$, $k=120$, $57 \leq c_3<\frac{1}{2}k=60$, $c_3 \mid k_2b_2=(120.119/10).110$, a contradiction.

If $b_{d-1}=6$, then $\mu\mu_\Delta=26$, $\mu=13$, $\mu_\Delta=2$ and $m=4$, contrary to our assumption.

If $b_{d-1}=4$, then $\mu\mu_\Delta=10$, so either $\mu=10$, $\mu_\Delta=1$, $m=4$, a contradiction, or $\mu=5$, $\mu_\Delta=2$, $m=2$, $k=25$, contradicting (7b).

Finally, if $b_{d-1}=2$, then $\mu_\Delta=1$, $\mu=2$, $m=0$, contrary to our assumption. □

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